Quantitative exponential bounds for the renewal theorem with spread-out distributions

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Abstract

We establish exponential convergence estimates for the renewal theorem in terms of a uniform component of the inter-arrival distribution, of its Laplace transform which is assumed finite on a positive interval, and of the Laplace transform of some related random variable. Although our bounds are not sharp, our approach provides tractable constructive estimates for the renewal theorem which are computable (theoretically and numerically, at least) for a general class of inter-arrival distributions. The proof uses a coupling, and relies on Lyapunov-Doeblin type arguments for some discrete time regenerative structure, which we associate with the renewal processes.

Keywords : renewal theorem, spread-out inter-arrivals, convergence rate, Lyapunov, coupling.

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1 Introduction and main statements

We consider a classic *renewal processes* $(T_n)_{n\geq 0}$ defined by $T_n = T_0 + \sum_{i=1}^n X_i$, with T_0 a given non-negative random variable, called *delay*, and $(X_i)_{i\geq 1}$ an independent sequence of i.i.d. random variables, equal in law to a given random variable $X > 0$ with finite mean. The random variables $(X_n)_{n\geq 1}$ and $(T_n)_{n\geq 0}$ are respectively called *inter-arrivals times* and *renewal instants* (or epochs). The *renewal measure* U, defined on $(0,\infty)$ by $U(dt) = \mathbb{E}\sum_{j=0}^\infty \delta_{T_j}(dt)$ (with δ_x the Dirac mass at $x > 0$), is the central object of study in renewal theory. In one of its simplest forms, the *Renewal Theorem* states that, asymptotically as a time parameter $t > 0$ goes to infinity, the renewal measure of an interval $(t, t+h]$ is proportional to $h > 0$, if the distribution of X is *non-arithmetic* (i.e. it is not supported on some real arithmetic sequence). More precisely, one has

$$U((t, t+h]) \to \frac{h}{\mu} \quad \text{as } t \to \infty, \tag{1}$$

where $0 < \mu = \mathbb{E}(X) < +\infty$. Originally established in the non-arithmetic case in [6] (and in [7] in the arithmetic case), the Renewal Theorem and its several proofs have motivated deep probabilistic

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ideas and developments. We refer to the work [8] for an analytic proof based on Choquet-Deny's Lemma and to [12] for the first probabilistic proof using coupling, both ultimately relying on the Hewitt-Savage Theorem. Self-contained probabilistic proofs were given in [16, 17, 14]. See also [13, 10, 2, 3, 4] and references therein for further background as well as for refinements or extensions of the Renewal Theorem, and p. 480 in [18] or the unpublished notes [1] for historical accounts.

It is well known that the tail of X qualitatively determines the asymptotic behavior of the renewal measure. For instance, if $\mathbb{E}(X^2) < \infty$ then the number of renewals in $(0,t]$ exhibits Gaussian fluctuations as t goes to infinity (see e.g. Prop. 6.3, Ch. V in [4]); if furthermore X has some finite exponential moment and is spread-out (see below for the context), the error in (1) is exponentially small (see e.g. Thm. 2.10, Ch. VII in [4]). However, besides some specific families of inter-arrival laws, the precise relation between the tail of X and the rate of convergence in the renewal theorem is only partially understood. For instance, in [5], sharp estimates were obtained in the arithmetic case, but only for monotone hazard rates. Some conditions relating the renewal convergence rate to the tail of X in the arithmetic case are discussed in [9]. Estimates in the spread-out case have been given in [16], [4] but they depend on asymptotic bounds on the renewal measure or equation.

The present note further explores the link between X and the speed of convergence in the renewal theorem by providing, for a wide class of inter-arrival distributions, tractable bounds which are computable (theoretically and numerically, at least) in terms of the law of X.

For the sake of concreteness, we will focus on inter-arrival distributions which have some finite exponential moment and we will furthermore assume they have a uniform component (which grants non-arithmeticity and allows for a simpler analysis). More precisely, introducing the notation $\mathcal{L}(\beta) := \mathbb{E}\big(e^{\beta X}\big)$, $\beta \in \mathbb{R}$ for the Laplace transform of X, we will make

Assumption 1 (exponential moment)**.** *The inter-arrival distribution admits some finite exponential moment:* $\exists\, \alpha > 0$ *such that* $\mathcal{L}(\alpha) < +\infty$.

We will also suppose that the law of X satisfies

Assumption 2 (uniform component)**.** *There exist* $c \geq L > 0$ *and* $\tilde{\eta} \in (0,1)$ *such that*

$$\mathbb{P}\big(X \in [t_1, t_2]\big) \geq \frac{\tilde{\eta}}{2L} \int_{t_1}^{t_2} \mathbf{1}_{[c-L,c+L]}(r)\, dr \qquad \text{for all } 0 \leq t_1 \leq t_2.$$

In other words, X *has a uniform component on the interval* $[c-L, c+L]$ *with mass* $\tilde{\eta}$.

In concrete examples, uniform components can usually be explicitly identified. Recall also that Assumption 2 holds for some convolution power of each spread-out distribution (i.e. one for which some convolution power has an absolutely continuous component, see Section VII.1 in [4]). Thus, our results also apply to spread-out distributions by considering some finite sum of the inter-arrival time lengths instead of a single one.

As in previous works, our approach will be based on a coupling argument, that is, on constructing on some probability space two copies of the renewal process with different initial delays, and estimating the tail of some random time at which they "coalesce". We briefly recall some general well known facts about such a construction (see [13, 18, 4] for more background) and then state our results.

Write $N_t := \sup\{n \in \mathbb{N} : T_n \leq t\} = \sum_{j=0}^{\infty} \mathbf{1}_{(0,t]}(T_j)$ for the total number of renewals until time $t \geq 0$ and denote the *residual life* (or forward recurrence time) process by

$$(B_t := T_{N_t+1} - t)_{t \geq 0}\,,$$

which is Markov. Let $(B'_t)_{t\geq 0}$, $(N'_t)_{t\geq 0}$ and U' denote the corresponding objects associated with a copy $(T'_n)_{n\geq 0}$ of the renewal process with same inter-arrival law, defined on the same probability

space as $(T_n)_{n\geq 0}$. An almost surely finite random time T^* such that a.s., $B_t = B'_t$ for all $t \geq T^*$, is called a *coupling time for* (B, B').

The recurrent process $(B_t)_{t\geq 0}$ has the stationary density $\mu^{-1}\mathbb{P}(X > t)dt$ and the renewal process with delay T_0 accordingly distributed is stationary (i.e. the corresponding renewal measure is equal to $\mu^{-1}dt$). The spread-out condition is necessary and sufficient for the residual life process to converge in total variation distance $\|\cdot\|_{TV}$ to its stationary distribution (see e.g. Cor.1.5 Ch.VII in [4]). By the coupling inequality (see [13]) one moreover has the estimate

$$\|law(B_t) - law(B'_t)\|_{TV} \leq \mathbb{P}(T^* > t). \tag{2}$$

Thus, finiteness of some exponential moment of T^* immediately grants exponential convergence to equilibrium at the same rate at least, by Chernoff's inequality. Moreover, since $N_{T^*+s} - N_{T^*} = N'_{T^*+s} - N'_{T^*}$ a.s. for all $s \geq 0$, for any given $t \geq 0$ one also gets the estimate

$$\left|U(t+D) - U'(t+D)\right| \leq \mathbb{E}\left(\mathbf{1}_{T^*>t}\left(\sum_{j=0}^{\infty}\mathbf{1}_{t+D}(T_j) + \sum_{j=0}^{\infty}\mathbf{1}_{t+D}(T'_j)\right)\right) \tag{3}$$

for all Borel sets $D \subset \mathbb{R}_+$. Our goal thus is to build two copies $(T_n)_{n\geq 0}$ and $(T'_n)_{n\geq 0}$ with a coupling time T^* having an exponential tail that can be explicitly controlled in terms of the law of X.

Let us introduce further notation required to state our results. In the sequel we write

$$\eta := \tilde{\eta}/2 \in (0, 1/2)$$

for the mass of the uniform component $[c, c+L]$ of X. We will also denote by $\bar{\mathcal{L}}_a : \mathbb{R} \mapsto \mathbb{R}_+ \cup \{\infty\}$ the Laplace transform of the maximum of two independent copies of the random variable X, both conditioned on being strictly larger than $a > 0$. Last, given $x \geq 0$, we denote by (B^x_t) the residual life process when $T_0 = x$ a.s. The following is our main result:

Theorem 1. *Suppose Assumptions 1. and 2. hold. Given $\beta \in \mathbb{R}_+, \delta \in [0,1)$ such that $\mathcal{L}((1+\delta)\beta) < \infty$, set*

$$R = R(\delta, \beta) := \frac{1}{2\beta}\log\left[\frac{\mathcal{L}((1+\delta)\beta)}{1 - \mathcal{L}(-(1-\delta)\beta)}\right].$$

For each $x \in \mathbb{R}_+$, there exists a coupling $(B^x_t, B^0_t)_{t\geq 0}$ with coupling time $T^*(x)$ such that

$$\mathbb{P}(T^*(x) > t) \leq \exp\{\theta\beta x\mathbf{1}_{x>R}\}\,\frac{\eta^{\lceil R/L\rceil}e^{\theta\beta\{R+\lfloor R/L\rfloor c\}}\bar{\mathcal{L}}_{c+L}(\theta\beta)}{1 - e^{\theta\beta\{R+\lfloor R/L\rfloor c\}}\bar{\mathcal{L}}_{c+L}(\theta\beta)\left(1-\eta^{\lceil R/L\rceil}\right)}\exp(-\theta\delta\beta t)$$

for every $t \geq x$ and all $\theta \in (0,1]$ for which $e^{\theta\beta\{R+\lfloor R/L\rfloor c\}}\bar{\mathcal{L}}_{c+L}(\theta\beta)\left(1-\eta^{\lceil R/L\rceil}\right) < 1$.

The above bound is involved, but can be better understood in terms of the parameters

$$a = (1+\delta)\beta \text{ and } b = (1-\delta)\beta.$$

Indeed, $R > 0$ will correspond to the smallest value for which we can grant that R-close renewals of two independent copies will occur within some random time lapse $T_R > 0$ of exponentially decaying length. This value is controlled by both a positive and a negative exponential moments of X, of orders a and $-b$ respectively, through the quantity

$$\left[\frac{\mathcal{L}(1+\delta)\beta)}{1 - \mathcal{L}(-(1-\delta)\beta))}\right]^{\frac{1}{2\beta}} = \left[\frac{\mathcal{L}(a)}{1-\mathcal{L}(-b)}\right]^{\frac{1}{a+b}}.$$

The random variable T_R will intervene a random number of times, geometrically distributed with parameter approximately equal to $\eta^{R/L}$. The exponential rate in Theorem 1 given by

$$\theta\delta\beta = \theta\frac{(a-b)}{2}$$

thus depends on the difference $(a-b)$, the Laplace transform $\hat{\mathcal{L}}(u) = \left[\frac{\mathcal{L}(a)}{1-\mathcal{L}(-b)}\right]^{\frac{u}{2}(1+\frac{c}{L})} \bar{\mathcal{L}}_{c+L}\left(\frac{u(a+b)}{2}\right)$ of some r.v. that accounts for the time cost of failing a coupling attempt, and some small enough $\theta \in (0,1)$ (so that $\hat{\mathcal{L}}(\theta)(1-\eta^{R/L}) < 1$) that controls the tradeoff between the previous ingredients. We then deduce

Corollary 1. *For each $\gamma < \beta\theta\delta$ as in Theorem 1, there is an explicit constant C depending on η, L, c and γ such that*

$$\|law(B_t^x) - law(B_t^0)\|_{TV} \leq \exp\{\beta\theta x\} C \exp(-\gamma t).$$

Moreover, if U^x and U^0 respectively denote the renewal measures associated with the processes $(B_t^x)_{t\geq 0}$ and $(B_t^0)_{t\geq 0}$, then for all Borel sets D in $(0,\infty)$ we have:

$$\left|U^x(t+D) - U^0(t+D)\right| \leq 2\exp\{\beta\theta x\} C \exp(-\gamma t)\,(U^0((0,\sup D)) + 1).$$

Remark 1. *By slight modifications of the proofs, it is also possible to replace the initial delay 0 by a generic one $y \neq x$. Moment conditions other than exponential can be treated with our techniques as well.*

In the next section, an outline of our approach and a plan of the proofs are presented. A comparison to previous coupling arguments together with a discussion of our results is given in **Section 3**.

2 Idea of the coupling and plan of the paper

Our coupling construction and estimates will rely on the discrete time structure of the renewal process. We start noting that, under Assumption 2, for any $s \in [0, L]$ and $t_2 \geq t_1 \geq s$ one has

$$\mathbb{P}\big(X + s \in [t_1, t_2]\big) \geq \frac{\eta}{L}\int_{t_1}^{t_2} \mathbf{1}_{[c-L+s,c+L+s]}(u)\,du \geq \frac{\eta}{L}\int_{t_1}^{t_2} \mathbf{1}_{[c,c+L]}(r)\,dr.$$

The random variables $(X+s)_{s\in[0,L]}$ thus have a common uniform component, of mass η, on the interval $[c, c+L]$. The following is a straightforward and useful consequence:

Lemma 1. *Under Assumption 2, for each $s \in [0, L]$ one can define, on some probability space, a Bernoulli r.v. ξ such that $\mathbb{P}(\xi = 1) = \eta = 1 - \mathbb{P}(\xi = 0)$, a uniform random variable U in $[c, c+L]$ independent of ξ, and random variables $X' \overset{d}{=} X$ and $X^{(s)} \overset{d}{=} X + s$, such that*

$$\begin{aligned} &\mathbb{P}(X' = X^{(s)} = U|\xi = 1) = 1\,,\\ &\qquad \mathbb{P}(X' \in dt|\xi = 0) = (1-\eta)^{-1}\left[\mathbb{P}(X \in dt) - \eta\mathbb{P}(U \in dt)\right] \forall\, t \geq 0,\\ &\quad \mathbb{P}(X^{(s)} \in dt|\xi = 0) = (1-\eta)^{-1}\left[\mathbb{P}(X + s \in dt) - \eta\mathbb{P}(U \in dt)\right] \forall\, t \geq 0, \quad \textit{with} \end{aligned} \tag{4}$$

X' and $X^{(s)}$ independent conditionally on $\{\xi = 0\}$. In particular, $(X', X'' := X^{(s)} - s)$ are two copies of the random variable X for which $X'' = X' - s$ holds with probability η.

Given a random variable Z, the same construction can be made conditionally on $\{Z = s\}$, in which case Lemma 1 holds true a.s. with respect to the law of Z, and X' and X'' are independent of Z (though the pair (X', X'') is not). Thus, starting from a relative initial delay of $x > 0$, by coupling $k = \lceil x/L \rceil$ pairs of consecutive inter-arrivals of the two processes, it is possible to produce simultaneous renewals with probability at least $\eta^{\lceil x/L \rceil} > 0$. However, this probability might be arbitrarily small if the initial relative delay x is not controlled, whereas, when such a "coupling attempt" fails, the resulting relative delay can in principle be arbitrarily large.

Our coupling will therefore consist in a two-steps iterative scheme. **Step 1** roughly consists in running two independent copies until renewals of both processes occur closer that some (large enough) $R > 0$. The following bounds for the time this requires will be proved in **Section 5**:

Proposition 1. *Given $R > 0$ and two independent copies $(T_n)_{n\geq 0}$ and $(T'_n)_{n\geq 0}$ of the renewal process such that $T_0 = 0$ and $T'_0 = x > 0$, let*

$$T_R = T_R(x) := \inf\{t \geq 0 : \exists n, n' \in \mathbb{N} \text{ such that } t = T_n \geq T'_{n'} - R \text{ or } t = T'_{n'} \geq T_n - R\}.$$

Then, if $R \geq \frac{1}{2\beta} \log\left[\frac{\mathcal{L}(\lambda+\beta)}{1-\mathcal{L}(-(\beta-\lambda))}\right]$, for all $0 < \lambda < \beta$ such that $\mathcal{L}(\lambda+\beta) < \infty$, we have

$$\mathbb{E}_x(e^{\lambda T_R}) \leq e^{\beta x \mathbf{1}_{x>R}} \quad \text{and} \quad \forall \lambda' \in [0, \lambda],\ \mathbb{E}_x(e^{\lambda' T_R}) \leq e^{\frac{\lambda'\beta}{\lambda} x \mathbf{1}_{x>R}}.$$

If we moreover write $D_R(x) = |T_n - T'_{n'}| \leq R$, for $n, n' \in \mathbb{N}$ such that $T_R(x) = T_n \geq T'_{n'} - R$ or $T_R(x) = T'_{n'} \geq T_n - R$, the process $(T_R(x), D_R(x) : x \geq 0)$ is measurable.

Although the coupling in Step 1 is a classic one, the previous exponential estimates are to our knowledge new; they rely on a Lyapunov-type argument for some discrete-time random walk defined in terms of the two copies' epochs. Notice that Step 1 is not run (i.e. $T_R(x) = 0$) if $x \leq R$.

As soon as the relative delay z between the two copies is less than R, **Step 2** puts in place the coupling suggested after Lemma 1. More precisely, in Step 2 we will use the coupling of two copies of the renewal process provided by the following result, which is proved in **Section 6**:

Lemma 2. *For each $z > 0$ one can define on some probability space two copies $(T_n)_{n\geq 0}$ and $(T'_n)_{n\geq 0}$ of the renewal process with $T_0 = 0$ and $T'_0 = z$ and a random variable $I \in \mathbb{N}$ a.s. bounded by $\lceil z/L \rceil$, such that the event $\Theta(z) := \{T_I = T'_I\}$ and the random variable $M(z) := \max\{T_I, T'_I\}$ satisfy, for each $R > 0$, the uniform bounds*

$$\inf_{z\in[0,R]} \mathbb{P}(\Theta(z)) \geq \eta^{\lceil R/L \rceil} > 0\,,$$

$$\sup_{z\in[0,R]} \mathbb{E}\left(e^{\gamma(M(z)-(z+c\lfloor z/L \rfloor))}\right) \leq \bar{\mathcal{L}}_{c+L}(\gamma)$$

and

$$\sup_{z\in[0,R]} \mathbb{E}\left(e^{\gamma(M(z)-(z+c\lfloor z/L \rfloor))}\mathbf{1}_{\Theta(z)^c}\right)(1-\eta^{\lceil z/L \rceil})^{-1} \leq \bar{\mathcal{L}}_{c+L}(\gamma)$$

for all $\gamma \in \mathbb{R}$. Moreover, setting $m(z) := \min\{T_I, T'_I\}$, this construction can be done simultaneously for all $z > 0$ in such a way that the process $(M(z), m(z), \mathbf{1}_{\Theta(z)} : z \geq 0)$ is measurable.

The random variable I in Lemma 2 will correspond to the minimal number of pairs of inter-arrivals, consecutively coupled as in Lemma 1, required to obtain simultaneous renewals with positive probability, if $T_0 = 0$ and $T'_0 = z$. Thus, if Step 2 is run after Step 1, the event $\Theta(z)$ will occur with uniformly lower bounded probability. We say in that case that the coupling succeeds;

otherwise, one goes back to Step 1 and iterates. The upper bounds in Lemma 2 moreover provide uniform exponential estimates of the continuous time $M(z)$ spent during one iteration of Step 2, whatever its outcome is, in terms of the relative delay z between the two copies at the beginning of it. Thus, even if their relative delay after Step 2 can be unbounded if the coupling attempt fails, these bounds will provide some control of the initial delay at the beginning of the next iteration of Step 1.

The proof of Theorem 1, given in **Section 4**, will consist in providing an exponential control of the total continuous time required by the two copies, constructed using this scheme, in order to have simultaneous renewals. Hence, it will bring together Proposition 1 and Lemma 2, by means of an exponential estimate on "sub-geometrical" sums of dependent positive random variables (Lemma 3 in **Appendix A.1**). The first statement of Corollary 1 is straightforward from Theorem 1 and inequality (2). The second one is more subtle and is proved in **Section 7**.

3 Comparison to previous couplings and discussion of our results

Proofs of renewal theorems given in [14] or [3], among others, rely on the hitting times of intervals $[-\varepsilon, \varepsilon]$ by a random walk $(T_n - T'_n)_{n\geq 0}$ defined in terms of two renewal processes with coupled inter-arrivals differing by less than $\varepsilon > 0$. Those random walks being symmetric, the expected number of steps in order that ε-close renewals occur and the expected real time required for that are infinite. Here, we will deal with a random walk which is strongly biased towards 0 and thus has some geometrically decaying hitting times. This walk is somehow reminiscent of a Markov chain studied in [16], but our arguments are quite different and avoid in particular the use of bounds based on the renewal equation. Our two-step coupling scheme is rather inspired by the celebrated Meyn-Tweedie approach to long time convergence of Markov processes (see [15]), but our discrete-time regenerative structure is different. Our strategy also differs from the regenerative process approach in continuous-time adopted in Ch. VII of [4], which at some point needs the use of asymptotic bounds on the renewal measure and hence cannot yield tractable estimates.

Another related reference is Chapter 6 in the book [10]. In this work, brought to the authors' attention by an anonymous referee, Kalashnikov develops techniques close to the ones of the present paper: he constructs a coupling under a condition of contraction in total variation (condition (3) stated p. 167) and then proves that this condition is satisfied when the inter-arrival times have a distribution in some class (given in Definition 2, p. 185), which is comparable to our Assumption 2. However, the author doesn't follow all the constants and this turns out to be a difficult task (in particular since Lemma 9 therein is given without proof). Thus, even if the ideas in [10] are close to ours, the present paper provides a more direct approach (without any abstract contraction condition), which makes it easier to exhibit bounds for the convergence rate.

Unfortunately, the joint dependance of our bounds on the parameters $(\beta, \theta, \delta) \in \mathbb{R}_+ \times (0,1)^2$ is not simple and, in particular, the optimization problem one needs to solve in order to maximize the convergence rate is not convex. Although its solution could be numerically approximated by some global optimization routine, while optimizing the uniform component considered as well, in general we do not expect to get sharp bounds, since our arguments rely on pessimistic (though careful) estimates. For instance, if S has the folded standard Gaussian distribution, a numerical optimization of our bounds yields the maximum rate $\beta^*\theta^*\delta^* = 0.001306$. In turn, a nonlinear regression fit on Monte-Carlo sample averages of quantities of the type $U^x(t+D)$ suggest in this case an exponential convergence rate about 3 orders of magnitude faster.

Nevertheless, the techniques here developed have the interest of providing tractable bounds for the renewal theorem in a general setting. In doing so they also give additional insight on the properties

of S involved in the speed of convergence. Our arguments could in principle be refined in order to take advantage of more specific features of the inter-arrivals (such as the additional integrability or increasing hazard rate of the above example). They should allow for extensions to more general frameworks in renewal theory as well.

4 Proof of Theorem 1

We start by estimating the total time spent during one iteration of Step 2 followed by one of Step 1 (in that order), when at the beginning of the former one of the two copies of the renewal process is $0-$delayed and the other one has delay $z > 0$. In the notations of Lemma 2, their relative delay at the end of Step 2 is $\Delta(z) := M(z) - m(z) \geq 0$, and one has $\Delta(z) = 0$ if the coupling is successful. The total time spent in one iteration of Steps 2 and then 1 has the same law as $T_R(\Delta(z)) + m(z)$, where $x \mapsto T_R(x)$ is a copy of the (measurable) process of Proposition 1, independent from the process $z \mapsto (\Delta(z), m(z), \mathbf{1}_{\Theta(z)})$. For fixed $0 < \lambda' < \lambda < \beta$ and $R \geq 0$ as in Proposition 1, we get

$$\begin{aligned}\mathbb{E}\left(\exp\{\lambda'(T_R(\Delta(z)) + m(z))\}\right) =& \mathbb{E}\left(\mathbb{E}\left(\exp\{\lambda'(T_R(x)\}\right)\Big|_{x=\Delta(z)} \exp\{\lambda' m(z)\}\right) \\ \leq& \mathbb{E}\left(\exp\left\{\frac{\lambda'\beta}{\lambda}\Delta(z) + \lambda' m(z)\right\}\right) \\ \leq& \mathbb{E}\left(\exp\left\{\frac{\lambda'\beta}{\lambda}M(z)\right\}\right).\end{aligned}$$

We similarly obtain

$$\begin{aligned}\mathbb{E}\left(\exp\{\lambda'(T_R(\Delta(z)) + m(z))\}\mathbf{1}_{\Theta(z)^c}\right) =& \mathbb{E}\left(\mathbb{E}\left(\exp\{\lambda'(T_R(x)\}\right)\Big|_{x=\Delta(z)} \exp\{\lambda' m(z)\}\mathbf{1}_{\Theta(z)^c}\right) \\ \leq& \mathbb{E}\left(\exp\left\{\frac{\lambda'\beta}{\lambda}M(z)\right\}\mathbf{1}_{\Theta(z)^c}\right).\end{aligned}$$

Now, by Lemma 2, $\inf_{z\in[0,R]} \mathbb{P}(\Theta(z)) \geq \eta^{\lceil R/L\rceil}$ for each $R > 0$. Taking $\gamma = \frac{\lambda'\beta}{\lambda}$ therein we get

$$\sup_{z\in[0,R]} \mathbb{E}\left(\exp\{\lambda'(T_R(\Delta(z)) + m(z))\}\right) \leq e^{\frac{\lambda'\beta}{\lambda}(R+\lfloor R/L\rfloor c)}\bar{\mathcal{L}}_{c+L}\left(\frac{\lambda'\beta}{\lambda}\right) \tag{5}$$

and

$$\sup_{z\in[0,R]} \mathbb{E}\left(\exp\{\lambda'(T_R(\Delta(z)) + m(z))\}\mathbf{1}_{\Theta(z)^c}\right) \leq e^{\frac{\lambda'\beta}{\lambda}(R+\lfloor R/L\rfloor c)}\bar{\mathcal{L}}_{c+L}\left(\frac{\lambda'\beta}{\lambda}\right)(1-\eta^{\lceil R/L\rceil}). \tag{6}$$

Let us now derive an exponential estimate for the global time required for the two copies in our coupling scheme to have simultaneous renewals. Thanks to the independence of the inter-arrivals of the renewal process and the measurability properties stated in Proposition 1 and Lemma 2, the relevant time-lengths in our coupling scheme can be constructed using independent sequences $((T_R^j(y), D_R^j(y)) : y \geq 0)_{j\in\mathbb{N}}$ and $((\Delta_j(z), m_j(z), \mathbf{1}_{\Theta_j(z)}) : z \geq 0)_{j\in\mathbb{N}\setminus\{0\}}$ of independent copies of the processes (T_R, D_R) and $(\Delta, m, \mathbf{1}_\Theta)$. More precisely, recursively defining

$$Y_0 = x, \quad Z_{j+1} = D_R^j(Y_j) \text{ and } Y_{j+1} = \Delta_{j+1}(Z_{j+1}), \quad j \geq 0,$$

the sequence $(Y_j, Z_{j+1})_{j\in\mathbb{N}}$ has the same law as the sequence of relative delays of the two copies, after the j−th iteration of Step 1 and the consecutive one of Step 2, respectively. A stochastic upper bound for the coalescing time of the two copies is then given by

$$\bar{T}^*(x) := T_R^0(x) + \sum_{j=1}^{\sigma} \left(T_R^j(Y_j) + m_j(Z_j) \right) ,$$

where $\sigma = \inf\{j > 0 \, : Y_j = 0\}$. Applying Lemma 3 in Appendix A.1 to the filtration $(\mathcal{G}_n)_{n\in\mathbb{N}}$, with

$$\mathcal{G}_n := \sigma\left(\left\{ T_R^j, D_R^j, \Delta_k, m_k, \mathbf{1}_{\Theta_k} : j = 0, \dots n \, , k = 1, \dots , n \right\} \right)$$

and the random variables and events $W_n = T_R^n(Y_n) + m_n(Z_n)$ and $A_n = \{\mathbf{1}_{\Theta_n(Z_n)} = 1\}$, we deduce, thanks to independence of the processes generating $\mathcal{G}_n$ and the bounds (5) and (6), that

$$\begin{aligned}
\mathbb{E}\left(e^{\lambda' \bar{T}^*(x)}\right) &\leq \mathbb{E}\left(e^{\lambda' T_R^0(x)}\right) \mathbb{E}\left(\left(e^{\left\{\frac{\lambda'\beta}{\lambda}(R+\lfloor R/L\rfloor c)\right\}} \bar{\mathcal{L}}_{c+L}\left(\frac{\lambda'\beta}{\lambda}\right)\right)^G \right) \\
&\leq e^{\frac{\lambda'\beta}{\lambda} x \mathbf{1}_{x>R}} \mathbb{E}\left(\left(e^{\left\{\frac{\lambda'\beta}{\lambda}(R+\lfloor R/L\rfloor c)\right\}} \bar{\mathcal{L}}_{c+L}\left(\frac{\lambda'\beta}{\lambda}\right)\right)^G \right)
\end{aligned}$$

where G is a geometric r.v. of parameter $\eta^{\lceil R/L \rceil} \in (0, 1)$. Given parameters β, θ and δ as in Theorem 1, its proof is then achieved by taking above $\lambda = \delta\beta$, $\lambda' = \theta\delta\beta$ and $R = R(\delta, \beta)$.

5 Step 1: a positive recurrent random walk associated with independent renewal processes

We next prove Proposition 1. We introduce to that end a biased random walk $(Y_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ defined from a single sequence of i.i.d. inter-arrivals $(X_n)_{n\in\mathbb{N}}$. The process $(Y_n)_{n\in\mathbb{N}}$ will account for the relative signed (positive or negative) delay of one fixed copy of the renewal process with respect to a second copy, after a total number n of inter-arrivals has occurred. More precisely, given an initial relative delay $x \in \mathbb{R}$, we set $Y_0 = x$. By convention, $x \geq 0$ means that one copy, henceforth fixed and called "the first copy", has a delay x, whereas the other copy, called "the second copy", is 0 delayed. Conversely, a relative initial delay $x < 0$ means that the first copy is 0 delayed and the second one has a delay $|x| > 0$. To construct the walk we proceed as follows: if for given n we have $Y_n \geq 0$, meaning that the first copy's last defined epoch occurred at distance Y_n to the right of the second copy's one, we add the next inter-arrival X_{n+1} to the last defined epoch of the second copy. If, on the contrary, we had $Y_n < 0$, this means that the first copy's last defined epoch occurred at distance $|Y_n|$ left from the second copy's one, and the random variable X_{n+1} is then added to the last defined epoch of the first copy. We thus have

$$Y_{n+1} = \begin{cases} Y_n - X_{n+1} & \text{if } Y_n \geq 0, \\ Y_n + X_{n+1} & \text{if } Y_n < 0. \end{cases}$$

Notice that the "leftmost copy" by the end of step n either catches up in step $n+1$ part of its delay with respect to the other copy or otherwise overshoots the lastly defined epoch of the latter, in which case the roles are then interchanged. Setting $N_n^+ := \inf\{m \in \mathbb{N} : \sum_{i=0}^m \mathbf{1}_{Y_{i-1}\geq 0} \geq n\}$ and $N_n^- := \inf\{m \in \mathbb{N} : \sum_{i=0}^m \mathbf{1}_{Y_{i-1}<0} \geq n\}$, it easily follows from the independence of the (X_n) that N_n^+ and N_n^- go to ∞ with n. Moreover, the inter-arrivals assigned to the first and second copies are respectively given by the sequences $(X_{N_n^+})_{n\geq 1}$ and $(X_{N_n^-})_{n\geq 1}$, and the strong Markov property

of the random walk $(Y_n)_{n\in\mathbb{N}}$ shows that these are independent i.i.d. sequences; they thus define independent copies of the renewal process.

The minimal total number of inter-arrivals required for epochs of these two processes to take place not farther that R from each other is

$$\tau_R := \inf\{n \geq 0 : |Y_n| \leq R\}.$$

Hence, in the notation of Proposition 1, we have

$$T_R(x) = \mathbf{1}_{|x|>R}\left(\sum_{i=1}^{\tau_R} X_i \mathbf{1}_{Y_{i-1}\geq 0} + Y_{\tau_R}\mathbf{1}_{Y_{\tau_R}<0}\right) \leq \bar{T}_R(x) := \mathbf{1}_{|x|>R}\sum_{i=1}^{\tau_R} X_i \mathbf{1}_{Y_{i-1}\geq 0},$$

and $D_R(x) = |Y_{\tau_R}|$. We will estimate exponential moments of $\bar{T}_R(x)$ by a Lyapunov-type argument. Let $0 \leq \lambda < \beta$ be such that $\mathcal{L}(\lambda+\beta) < \infty$ and set $V(x) := e^{\beta|x|}$. Then,

$$\begin{aligned}
&\mathbb{E}_x\Big(V(Y_1)e^{\lambda X_1 \mathbf{1}_{x\geq 0}}\Big)\\
&\quad\leq e^{-\beta|x|}\Big[\mathbf{1}_{x\geq 0}\mathbb{E}(e^{(\lambda+\beta)X}) + \mathbf{1}_{x<0}\mathbb{E}(e^{\beta X})\Big] + e^{\beta|x|}\Big[\mathbf{1}_{x<0}\mathbb{E}(e^{-\beta X}) + \mathbf{1}_{x\geq 0}\mathbb{E}(e^{(\lambda-\beta)X})\Big]\\
&\quad\leq e^{-\beta|x|}\mathbb{E}(e^{(\lambda+\beta)X}) + e^{\beta|x|}\mathbb{E}(e^{(\lambda-\beta)X})\\
&\quad\leq V(x)\Big(\mathcal{L}(-(\beta-\lambda)) + e^{-2\beta R}\mathcal{L}(\lambda+\beta)\Big) + \mathcal{L}(\lambda+\beta)\mathbf{1}_{[0,R]}(|x|)
\end{aligned}$$

where the first inequality is obtained after partitioning the expectation according to the signs of $x - X_1$ and of x. By standard arguments, the above bound entails that the discrete time process

$$V(Y_{\tau_R\wedge n})e^{\lambda\sum_{i=1}^{\tau_R\wedge n} X_i\mathbf{1}_{\{Y_{i-1}\geq 0\}}}\rho_{\beta,\lambda,R}^{-\tau_R\wedge n}, \quad n\in\mathbb{N},$$

with

$$\rho_{\beta,\lambda,R} := \mathcal{L}(-(\beta-\lambda)) + e^{-2\beta R}\mathcal{L}(\lambda+\beta),$$

is a supermartingale in the discrete filtration generated by the sequence $(X_i)_{i\geq 1}$. In particular,

$$\mathbb{E}_x\left(e^{\lambda\sum_{i=1}^{\tau_R\wedge n} X_i\mathbf{1}_{Y_{i-1}\geq 0}}\rho_{\beta,\lambda,R}^{-\tau_R\wedge n}\right) \leq \mathbb{E}_x\left(V(Y_{\tau_R\wedge n})e^{\lambda\sum_{i=1}^{\tau_R\wedge n} X_i\mathbf{1}_{Y_{i-1}\geq 0}}\rho_{\beta,\lambda,R}^{-\tau_R\wedge n}\right) \leq V(x).$$

By letting $n\to\infty$ in the first expectation above, we deduce that $\tau_R < \infty$ a.s. if $\rho_{\beta,\lambda,R}^{-1} > 1$ or if $\rho_{\beta,\lambda,R}^{-1} = 1$ and $0 < \lambda < \beta$ (in the second case we use the fact that $\sum_{i=1}^n X_i\mathbf{1}_{Y_{i-1}\geq 0} = \sum_{i=1}^{n^+} X_{N_i^+}$ is a sum of i.i.d. random variables). This yields

$$\mathbb{E}_x\left(e^{\lambda\bar{T}_R(x)}\right) \leq e^{\beta x\mathbf{1}_{x>R}}$$

whenever $\rho_{\beta,\lambda,R} \leq 1$ or, equivalently, when $R \geq \frac{1}{2\beta}\log\left[\frac{\mathcal{L}(\lambda+\beta)}{1-\mathcal{L}(-(\beta-\lambda))}\right] = R(\lambda/\beta,\beta)$. The first assertion of Proposition 1 follows. The second one is easily obtained with Holder's inequality. The last assertion of Proposition 1 is straightforward from the previous construction.

6 Step 2: attempting an exact coupling

We first construct the coupling of Lemma 2, in such a way that the measurability condition in its last assertion is granted from the beginning; we then establish the claimed exponential estimates.

Consider the four independent i.i.d. sequences: $(\xi_i)_{i=1}^{\infty}$ of Bernoulli r.v. with $\mathbb{P}(\xi_i = 1) = \eta = 1 - \mathbb{P}(\xi_i = 0)$, $(U_i)_{i=1}^{\infty}$ of uniform r.v. in $[c, c+L]$, and $(W_i')_{i=1}^{\infty}$ and $(\hat{W}_i)_{i=1}^{\infty}$ of r.v. such that

$$\begin{aligned}
&\mathbb{P}(W_i' \in dt) = (1-\eta)^{-1}\left[\mathbb{P}(X \in dt) - \eta\mathbb{P}(U \in dt)\right],\, t \geq 0 \text{ and}\\
&\mathbb{P}(\hat{W}_i \in dt) = (1-\eta)^{-1}\left[\mathbb{P}(X + L \in dt) - \eta\mathbb{P}(U \in dt)\right],\, t \geq 0,
\end{aligned}$$

with U uniformly distributed in $[c, c+L]$. Consider also ϑ a uniform random variable in $[0,1]$ independent of all the previous ones. By Lemma 3.22 in [11] there exists a measurable function $\Phi : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ such that, for each $z \in \mathbb{R}_+$, the random variable $\Phi(z, \vartheta)$ satisfies

$$\mathbb{P}(\Phi(z,\vartheta) \in dt) = (1-\eta)^{-1}\left[\mathbb{P}(X + (z - L\lfloor z/L \rfloor) \in dt) - \eta\mathbb{P}(U \in dt)\right],t \geq 0.$$

Set now $k = k(z) := \lceil z/L \rceil$ and for $i \geq 1$ define:

$$\begin{aligned}
X_i' &:= \mathbf{1}_{\xi_i=1}U_i + \mathbf{1}_{\xi_i=0}W_i',\\
\hat{X}_i &:= \mathbf{1}_{\xi_i=1}U_i + \mathbf{1}_{\xi_i=0}\left(\mathbf{1}_{i<k(z)}\hat{W}_i + \mathbf{1}_{i=k(z)}\Phi(z,\vartheta)\right) \text{ and}\\
X_i'' &:= \hat{X}_i - \left(\mathbf{1}_{i<k(z)}L + \mathbf{1}_{i=k(z)}(z - L\lfloor z/L \rfloor)\right).
\end{aligned}$$

We remark for later use that the r.v. W_i'' defined as $W_i'' := \hat{W}_i - L$, for $1 \leq i < k$, and as $W_k'' := \Phi(z,\vartheta) - (z - L\lfloor z/L \rfloor)$, all have the same law as the r.v. W_i'.
The sequences $(X_j')_{j=1}^{k}$ and $(X_j'')_{j=1}^{k}$ are both i.i.d. with the same law as X, they are measurable functions jointly in z and randomness and, on $F_k := \{(\xi_1, \ldots, \xi_k) = (1, \ldots, 1)\}$, it a.s. holds that

$$X_1' + \cdots + X_k' - (X_1'' + \cdots + X_k'') = z.$$

In particular, the probability of having such an equality is bounded from below by $\eta^{\lceil z/L \rceil} > 0$. The coupling will then consist in sampling the random variables ξ_i, X_i' and X_i'' up to the random index

$$I := \inf\{j \geq 0 : \xi_j = 0\} \wedge k \leq \lceil z/L \rceil.$$

If the latter set is empty, then the event F_k occurs, simultaneous renewals take place at time $X_1' + \cdots + X_k'$ and the coupling attempt is successful; otherwise, we say that it fails. Notice that if $1 \leq I < k$, the coupling attempt is said to fail, even if the I-th renewals of the two copies take place simultaneously (which can for instance happen if X has atoms). Notice also that when $I = k$, the coupling might succeed or fail. In all cases, we have

$$M(z) = \max\left\{\sum_{j=1}^{I} X_j', z + \sum_{j=1}^{I} X_j''\right\}.$$

The random variable $m(z)$ in the statement corresponds to $m(z) = \min\left\{\sum_{j=1}^{I} X_j', z + \sum_{j=1}^{I} X_j''\right\}$, and the event $\Theta(z)$, which corresponds to $\{M(z) = m(z)\}$, occurs if F_k does. The indicator function in the second estimate in Lemma 2 can thus be replaced by $\mathbf{1}_{F_k^c}$. Since $X_j'' \leq c$ for $j \leq I$ and $I \leq \lfloor z/L \rfloor$, we always have (with $\sum_{\emptyset} = 0$)

$$\sum_{j=1}^{I-1} X_j' \leq z + \sum_{j=1}^{I-1} X_j'' \leq z + c\lfloor z/L \rfloor.$$

Moreover, on F_k we have $X'_I, X''_I \leq (c+L)$. It then follows on one hand that, for all $\gamma \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\left(e^{\gamma M(z)}\right) \leq & e^{\gamma(z+c\lfloor z/L\rfloor)}\left[e^{\gamma(c+L)}\mathbb{P}(F_k) + \mathbb{E}\left(e^{\gamma\max\{X'_I,X''_I\}}|F_k^c\right)\mathbb{P}(F_k^c)\right]\\
\leq & e^{\gamma(z+c\lfloor z/L\rfloor)}\max\left\{e^{\gamma(c+L)}, \mathbb{E}\left(e^{\gamma\max\{X'_I,X''_I\}}|F_k^c\right)\right\}\\
\leq & e^{\gamma(z+c\lfloor z/L\rfloor)}\mathbb{E}\left(e^{\gamma\max\{(c+L),X'_I,X''_I\}}|F_k^c\right) \leq \infty.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
\mathbb{E}\left(e^{\gamma M(z)}\mathbf{1}_{F_k^c}\right) \leq & e^{\gamma(z+c\lfloor z/L\rfloor)}\mathbb{E}\left(e^{\gamma\max\{X'_I,X''_I\}}|F_k^c\right)\mathbb{P}(F_k^c)\\
\leq & e^{\gamma(z+c\lfloor z/L\rfloor)}\mathbb{E}\left(e^{\gamma\max\{(c+L),X'_I,X''_I\}}|F_k^c\right)\mathbb{P}(F_k^c) \leq \infty.
\end{aligned}$$

The two required estimates will then be proved by showing that

$$\mathbb{E}\left(e^{\gamma\max\{(c+L),X'_I,X''_I\}}|F_k^c\right) \leq \mathbb{E}\left(e^{\gamma\max\{W',W''\}}\right), \tag{7}$$

for independent r.v. (W', W'') of law $\left(\frac{\mathbb{P}(X\leq c+L)-\eta}{1-\eta}\right)\delta_{c+L}(ds) + \frac{\mathbf{1}_{s>c+L}}{1-\eta}\mathbb{P}_X(ds)$ with $\mathbb{P}_X$ the law of X. Indeed, since $\mathbb{P}(X > c+L) \leq 1-\eta$, one gets $\mathbb{P}(W' > s) \leq \mathbb{P}(X > s|X > c+L)$ for all $s \geq 0$, that is, W' is stochastically smaller than a r.v. $\bar{X}'$ equal in law to X conditioned on being not smaller than $c+L$. It then follows that $\max\{W', W''\}$ is stochastically smaller than $\max\{\bar{X}', \bar{X}''\}$ for an i.i.d. pair $(\bar{X}', \bar{X}'')$, from where we conclude.

Let us thus check inequality (7). Since $\mathbf{1}_{F_k^c} = \sum_{l=1}^{k-1}\mathbf{1}_{I=l} + \mathbf{1}_{I=k,\xi_k=0}$, we have

$$\begin{aligned}
\mathbb{E}\left(e^{\gamma\max\{(c+L),X'_I,X''_I\}}\mathbf{1}_{F_k^c}\right) = & \sum_{l=1}^{k-1}\mathbb{P}(I=l)\mathbb{E}\left(e^{\gamma\max\{(c+L),W'_l,W''_l\}}\right)\\
& + \mathbb{P}(I=k,\xi_k=0)\mathbb{E}\left(e^{\gamma\max\{(c+L),W'_k,W''_k\}}\right)
\end{aligned}$$

so it suffices to show that, for $l = 1,\dots,k$, $\mathbb{E}\left(e^{\gamma\max\{(c+L),W'_l,W''_l\}}\right)$ is bounded by $\mathbb{E}\left(e^{\gamma\max\{W',W''\}}\right)$. This follows from

$$\begin{aligned}
\mathbb{E}\left(e^{\gamma\max\{(c+L),W'_l,W''_l\}}\right) = & e^{\gamma(c+L)}\mathbb{E}\left(\mathbf{1}_{W'_l,W''_l\leq c+L}\right) + \mathbb{E}\left(e^{\gamma W'_l}\mathbf{1}_{W'_l>c+L\geq W''_l}\right)\\
& + \mathbb{E}\left(e^{\gamma W''_l}\mathbf{1}_{W''_l>c+L\geq W'_l}\right) + \mathbb{E}\left(e^{\gamma\max\{W'_l,W''_l\}}\mathbf{1}_{\min\{W'_l,W''_l\}>c+L}\right).
\end{aligned}$$

and the fact that, for each $l = 1,\dots k-1$, (W'_l, W''_l) are independent, $\mathbb{P}(W'_l \leq c+L) = \mathbb{P}(W''_l \leq c+L) = \frac{\mathbb{P}(X\leq c+L)-\eta}{1-\eta}$ and $\mathbb{E}(f(W'_l)\mathbf{1}_{W'_l>c+L}) = \mathbb{E}(f(W''_l)\mathbf{1}_{W''_l>c+L}) = \frac{\mathbb{E}(f(X)\mathbf{1}_{X>c+L})}{1-\eta}$ for all nonnegative measurable function f.

7 Bounds for the renewal measure

Thanks to inequality (3) and the fact that $t + D \subseteq (t, t+h]$ for $h = \sup D$, to prove the second statement of Corollary 1 it is enough to show that, for any $h > 0$,

$$\mathbb{E}\left(\mathbf{1}_{T^*(x)>t}\left(\sum_{j=0}^{\infty}\mathbf{1}_{(t,t+h]}(T''_j)\right)\right) \leq \mathbb{P}(T^*(x) > t)(U^0((0,h]) + 1) \tag{8}$$

for $(T''_n) = (T_n)$ and $(T''_n) = (T'_n)$ the epochs of the two copies. To that end we describe the discrete time structure used in constructing our coupling in a slightly different way from before. Consider the following independent i.i.d. sequences:

- $(\tilde{X}_k)_{k=1}^{\infty}$ with law equal to that of X,
- $(U_k)_{k=1}^{\infty}$ uniform in $[c, c+L]$,
- $(\xi_k)_{k=1}^{\infty}$ Bernoulli of parameter η,
- $(W'_k)_{k=1}^{\infty}$ and $(\hat{W}_k)_{k=1}^{\infty}$ with the laws described in Section 6 and
- $(\vartheta_k)_{k=1}^{\infty}$ uniform in $[0,1]$.

We can then construct our coupling using the i.i.d. random vectors $(\tilde{X}_k, U_k, \xi_k, W'_k, \hat{W}_k, \vartheta_k)_{k\in\mathbb{N}}$, as follows. We run Step 1 using the random variables $(\tilde{X}_k)$ to construct the random walk of Section 5, until the conditions required to start Step 2 (i.e. a relative delay not larger that R) are met. This first happens at some discrete random time, which is a stopping time with respect to the filtration $(\mathcal{F}_m)_{m\geq 1}$ defined by

$$\mathcal{F}_m := \sigma\left(\tilde{X}_k, U_k, \xi_k, W'_k, \hat{W}_k, \vartheta_k : k = 1, \dots, m\right).$$

Notice that, until then, the remaining coordinates $(U_k, \xi_k, W'_k, \hat{W}_k, \vartheta_k)$ of the vector are not used. Moreover, one and only one copy of the renewal process has had a renewal at each time step k. Right after that stopping time, we start Step 2 using at each time step k some random variables among $U_k, \xi_k, W'_k, \hat{W}_k$ and ϑ_k (as needed in the scheme described in Section 6). This is done until some second stopping time (with respect to $(\mathcal{F}_m)_{m\geq 1}$) at which the coupling attempt succeeds or fails. In the latter case one restarts Step 1. Notice that during Step 2, both copies have one renewal at each time step k.

We denote by $\tau(1) < \tau(2) < \tau(3) < \cdots$ (resp. $\tau'(1) < \tau'(2) < \tau'(3) < \cdots$) the discrete times k at which the number of arrivals of the first (resp. second) copy of the renewal process is increased by one additional unit. Notice they are also stopping times with respect to $(\mathcal{F}_m)_{m\geq 1}$.

We then denote by X_n (resp. X'_n) the increment of the first (resp. second) copy at time $k = \tau(n)$ (resp. $k = \tau'(n)$). It is then not hard to see that the sequence $(X_n, X'_n)_{n\geq 1}$ has the same law as the sequence of pairs of inter-arrivals resulting from our coupling construction. Moreover, $(X_n)_{n\geq 1}$ and $(X'_n)_{n\geq 1}$ are respectively adapted to the filtrations $(\mathcal{F}_{\tau(n)})_{n\geq 1}$ and $(\mathcal{F}_{\tau'(n)})_{n\geq 1}$.

Denote by (T_n) and N_t the epochs and counting processes corresponding to this sequence (X_n) and observe that $\{N_t = n\} = \{T_n \leq t\} \cap \{T_{n+1} > t\} \in \mathcal{F}_{\tau(n+1)}$. Moreover, $\{T^*(x) > t, N_t = n\} \in \mathcal{F}_{\tau(n+1)}$ since this event can be written in terms of $\{N_t = n\}$ and the family of random variables $\left(\mathbf{1}_{\{k\leq\tau(n+1)\}}(\tilde{X}_k, U_k, \xi_k, W'_k, \hat{W}_k, \vartheta_k)\right)_{k\geq 1}$

Defining now a function F on $[0,\infty)^{\mathbb{N}\setminus\{0\}}$ by $F(x_1, x_2, \dots) = \sum_{j=1}^{\infty} \mathbf{1}_{(0,h]}(\sum_{k=1}^{j}(x_k))$, the expectation in the left hand side of (8) is seen to be equal to

$$\begin{aligned}\sum_{n\in\mathbb{N}} \mathbb{E}\left(\mathbf{1}_{\{T^*(x)>t, N_t=n\}}\Big(\sum_{j=n+1}^{\infty} \mathbf{1}_{(t,t+h]}(T_j)\Big)\right)& \\ = \sum_{n\in\mathbb{N}} \mathbb{E}\left(\mathbf{1}_{\{T^*(x)>t, N_t=n\}} F(T_{n+1} - t, X_{n+2}, X_{n+3}, \dots)\right)& \\ \leq \sum_{n\in\mathbb{N}} \mathbb{E}\left(\mathbf{1}_{\{T^*(x)>t, N_t=n\}} (1 + F(X_{n+2}, X_{n+3}, \dots))\right).&\end{aligned}$$

To conclude (8) for $(T''_n) = (T_n)$ it suffices to check that $\mathbb{E}\left(\mathbf{1}_{\{T^*(x)>t, N_t=n\}} F(X_{n+2}, X_{n+3}, \dots)\right) = \mathbb{P}\{T^*(x) > t, N_t = n\}\mathbb{E}\left(F(X_1, X_2, ...)\right)$ for all $n \in \mathbb{N}$. This property is a consequence of the strong Markov property of the (i.i.d.) process $\left(\tilde{X}_k, U_k, \xi_k, W'_k, \hat{W}_k, \vartheta_k\right)_{k\geq 1}$ since, for each $n \in \mathbb{N}$,

$\{T^*(x) > t, N_t = n\} \in \mathcal{F}_{\tau(n+1)}$ and the r.v. $X_{n+2}, X_{n+3}, \ldots$ can be constructed using the random vectors $\left(\mathbf{1}_{\{k\geq\tau(n+2)\}}(\tilde{X}_k, U_k, \xi_k, W'_k, \hat{W}_k, \vartheta_k)\right)_{k\geq 1}$. The proof for $(T''_n) = (T'_n)$ is similar.

A Appendix

A.1 Laplace bounds for sub-geometric sums of dependent random variables

Lemma 3. *Let $(\mathcal{G}_n)_{n\geq 0}$ be a filtration, and $(A_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ sequences of respectively adapted events and adapted nonnegative random variables. Let $\sigma := \inf\{n \geq 1 : \mathbf{1}_{A_n} = 1\}$ and assume there exist $p \in (0,1)$ and a function $\psi : I \to \mathbb{R}$ defined in some real interval I, such that for all $\lambda \in I$,*

i) $\mathbb{E}\left(e^{\lambda W_n}|\mathcal{G}_{n-1}\right) \leq e^{\psi(\lambda)}$ a.s. on $A_1^c \cap \cdots \cap A_{n-1}^c$ if $n > 1$ and a.s. if $n = 1$ and

ii) $\mathbb{E}\left(e^{\lambda W_n}\mathbf{1}_{A_n^c}|\mathcal{G}_{n-1}\right) \leq (1-p)e^{\psi(\lambda)}$ a.s. on $A_1^c \cap \cdots \cap A_{n-1}^c$ if $n > 1$.

Then, $\mathbb{E}\left(e^{\lambda\sum_{j=1}^{\sigma} W_j}\right) \leq \mathbb{E}\left(\left(e^{\psi(\lambda)}\right)^G\right)$ for all $\lambda \in I$, where G is geometric of parameter p. In particular, if $0 \in I$ and ψ is increasing and goes to 0 at 0, we have $\mathbb{E}\left(e^{\lambda\sum_{j=1}^{\sigma} W_j}\right) \leq \frac{pe^{\psi(\lambda)}}{1-e^{\psi(\lambda)}(1-p)}$ for all λ such that $\psi(\lambda) < -\log(1-p)$.

If $\psi(0) = 0$, condition ii) classically yields that σ is stochastically smaller than G (see e.g. Lemma A.6 in [4]). The bound in Lemma 3 is sharp given the assumptions (it is attained for (W_n) i.i.d. of exponential law with Laplace transform $e^{\psi(\lambda)}$ and $\sigma = G$ independent).

Proof . We may assume that $\psi(\lambda)$ is in the domain of the Laplace transform of G. Moreover, replacing (A_n) by $(\tilde{A}_n)$ defined as $\tilde{A}_n = A_n$ if $n \leq N$ and $\tilde{A}_n = \Omega$ if $n \geq N+1$, for some fixed integer N, we may assume that $\sigma = \sigma \wedge N \leq N$ and then pass to the general case using monotone convergence. The fact that σ is bounded justifies the interchange of sums with differences needed to get (with the convention $\sum_{j=1}^{0} = 0$) :

$$\begin{aligned}\mathbb{E}\left(e^{\lambda\sum_{j=1}^{\sigma} W_j} - 1\right) &= \sum_{n=1}^{\infty} \mathbb{E}\left(\left[e^{\lambda\sum_{j=1}^{n} W_j} - e^{\lambda\sum_{j=1}^{n-1} W_j}\right]\mathbf{1}_{\sigma\geq n}\right) \\ &= \sum_{n=1}^{\infty} \mathbb{E}\left(e^{\lambda\sum_{j=1}^{n-1} W_j}\mathbb{E}\left(e^{\lambda W_n} - 1|\mathcal{G}_{n-1}\right)\mathbf{1}_{\sigma\geq n}\right).\end{aligned}$$

We deduce, using i) to get the first inequality and ii) to get the second one, that

$$\begin{aligned}&\mathbb{E}\Big(e^{\lambda\sum_{j=1}^{\sigma} W_j} - 1\Big) \\ &\quad\leq \sum_{n=1}^{\infty}\left[e^{\psi(\lambda)} - 1\right]\mathbb{E}\left(\mathbf{1}_{n=1} + e^{\lambda\sum_{j=1}^{n-1} W_j}\mathbf{1}_{A_{n-1}^c}\cdots\mathbf{1}_{A_1^c}\mathbf{1}_{n\geq 2}\right) \\ &\quad= \sum_{n=1}^{\infty}\left[e^{\psi(\lambda)} - 1\right]\mathbb{E}\left(\mathbf{1}_{n=1} + \mathbb{E}\left(e^{\lambda W_{n-1}}\mathbf{1}_{A_{n-1}^c}|\mathcal{G}_{n-2}\right)\left(\mathbf{1}_{n=2} + e^{\lambda\sum_{j=1}^{n-2} W_j}\mathbf{1}_{A_{n-2}^c}\cdots\mathbf{1}_{A_1^c}\mathbf{1}_{n\geq 3}\right)\right) \\ &\quad\leq \sum_{n=1}^{\infty}\left[e^{\psi(\lambda)} - 1\right]\mathbf{1}_{n=1} + \left[e^{2\psi(\lambda)} - e^{\psi(\lambda)}\right](1-p)\,\mathbb{E}\left(\mathbf{1}_{n=2} + e^{\lambda\sum_{j=1}^{n-2} W_j}\mathbf{1}_{A_{n-2}^c}\cdots\mathbf{1}_{A_1^c}\mathbf{1}_{n\geq 3}\right).\end{aligned}$$

Conditioning on $\mathcal{G}_{n-3}$ in the last expectation and iterating the argument yields the upper bound

$$\sum_{n=1}^{\infty}\left[e^{n\psi(\lambda)} - e^{(n-1)\psi(\lambda)}\right](1-p)^{n-1} = \sum_{n=1}^{\infty}\left[e^{n\psi(\lambda)} - e^{(n-1)\psi(\lambda)}\right]\mathbb{P}(G \geq n) = \mathbb{E}\left(\left(e^{\psi(\lambda)}\right)^G - 1\right).$$

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